Lesson 2-1 The Language of Functions

Vocabulary

mathematical model
relation
independent variable
dependent variable
function, ordered pair definition
domain of a function
range of a function
function, correspondence definition
real function
member of a set, element of a set, $\in$
piecewise definition of a function
value of a function

BIG IDEA A function is a special type of relation that can be described by ordered pairs, graphs, written rules or algebraic rules such as equations.

On pages 78 and 79, nine ordered pairs of numbers are listed and graphed. The first coordinate is the number of years after 1900; the second is the U.S. national debt (in billions of dollars) for that year. Any set of ordered pairs is a **relation**. In many contexts, the second number in each ordered pair depends in some way on the first number. When this happens, the first variable in a relation is called the **independent variable** and the second variable is called the **dependent variable**. For the national debt, the number x of years after 1900 is the independent variable and the debt y in that year is the dependent variable.

Mental Math

Write each expression as a power of x.

a. $(x^{50})^3$

b. $x^{50} \cdot x^{-22}$

c. $\frac{x^{50}}{x^{53}}$

What Is a Function?

When each value of the independent variable determines exactly one value of the dependent variable, the relation is called a *function*.

Ordered Pair Definition of Function

A **function** is a set of ordered pairs (x, y) in which each first component (x) is paired with exactly one second component (y).

For example, $f = \{(1, 2), (2, 4), (3, 7)\}$ is a function, but

$g = \{(1, 2), (2, 4), (1, 7)\}$ is not a function because 1 is paired with both 2 and 7.

In the set of ordered pairs of a function, the set of first components is the **domain** of the function. The set of second components is the **range** of the function. The domain consists of all allowable values of the independent variable; the range is the set of possible values for the dependent variable. The domain of the function f above is $\{1, 2, 3\}$, and the range is $\{2, 4, 7\}$.

STOP QY1

QY1

Give the domain and range of the function $h = \{(5, 1), (6, 3), (7, 1), (8, 3)\}$.

For the national debt data on page 78, we can say, "The U.S. national debt is a function of the year." The domain is the set of all years the U.S. has had and will have a national debt; the range is the set of all amounts of the national debt at the end of those years.

Another definition of *function* stresses the correspondence between the independent and dependent variables.

Correspondence Definition of Function

A **function** is a correspondence between two sets A and B in which each element of A corresponds to exactly one element of B.

The domain is the set A of values of the *independent variable*. The range is the set of only those elements of B that correspond to elements in A; these are the values of the *dependent variable*.

For most functions studied in this course, A and B are sets of real numbers. Functions whose domain and range are sets of real numbers are called **real functions.** Unless the domain of a function is explicitly stated, you may assume that it is the set of all real numbers for which the function is defined. Real functions have the useful characteristic that they can be pictured by coordinate graphs.

We use the symbols in the table at the right to represent sets of numbers. "*z*" stands for the German verb *zahlen,* "to count," and "*q*" stands for "quotient." These symbols can also be used in descriptions of other sets. If x is in a set A, then x is said to be a **member** or **element** of A, written $x \in A$. For instance, $3\frac{1}{2} \in \mathbb{Q}$. Similarly, every even integer can be written as $2 \cdot n$, where n is in the set of all integers. So we can write the set of even integers as $\{2n \mid n \in \mathbb{Z}\}$, read "the set of $2n$ such that n is an integer."

The symbol	represents the set of all
$\mathbb{Z}$	integers.
$\mathbb{R}$	real numbers.
$\mathbb{Q}$	rational numbers.
$\mathbb{N}$	natural numbers.

Example 1

A bakery charges $2.00 per muffin. Customers get a $2.00 discount for every 6 muffins purchased.

a Which statement is true: "the cost *c* is a function of the number *m* of muffins" or "the number *m* of muffins is a function of the cost *c*?"

b. Identify the independent and dependent variables of the function.

c. State the domain and range of the function.

Solution

a. Because there is exactly one cost c for a given number of muffins, the cost is a function of the number of muffins. A customer who buys 5 muffins pays the same amount as one who buys 6 muffins, so the number of muffins is not a function of the cost.

b. Because c depends on m, m is the independent variable and c is the dependent variable.

c. The domain is the set of all possible values of m. Because "negative muffins" does not make sense and you cannot buy part of a muffin, the domain is the set of nonnegative integers, which can be written $\{m \mid m \in \mathbb{Z} \text{ and } m \geq 0\}$. The range is the set of all possible values of c. Any even-number cost in dollars is possible, so the range is the set of nonnegative even integers.

Descriptions of Functions

Functions can be described in many ways. Some frequently-used descriptions are (1) tables or lists of ordered pairs, (2) rules expressed in words or equations, and (3) coordinate graphs. You should know how to recognize functions described in each of these ways, and how to convert from one description to another.

GUIDED

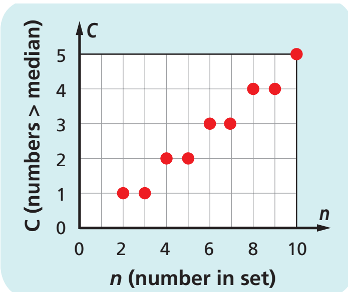
Example 2

Let n be an integer with $n \geq 2$. Graph the function that shows how many elements of an ordered set of n different integers are greater than the median of that set.

Solution Copy and complete a table similar to the one below with possible sets for the values of n from 2 to 10. Circle and count the elements greater than the median. Call this count C. Make a scatterplot of the points (n, C) for n from 2 to 10.

n	Sample set (ordered)	Count C of numbers greater than the median
2	{2, (5)}	1
3	{24, _?_, (68)}	1
4	{13, _?_, (35, _?_)}	_?_
5	{_?_, _?_, _?_, (_?_, _?_)}	_?_
⋮	⋮	⋮

Each ordered pair of this function is of the form (n, C) where n is the number of elements in the set and C is the number of elements greater than the median. So for integer values of n from 2 to 10, the ordered pairs of the function were found in the table: (2, 1), (3, 1), (4, _?_), (5, _?_), (6, _?_), (7, _?_), (8, _?_), (9, _?_), and (10, _?_). The ordered pairs are graphed at the right. Notice that we do not connect the dots, because the values of n are *discrete*. Also notice that pairs of n-values share the same count or frequency, C.



STOP **QY2**

In Example 2, the function is described in a table and a graph. Another description of the function in Example 2 combines two rules, one when n is even, and the other when n is odd.

$$C = \begin{cases} \frac{n}{2}, & \text{when } n \text{ is even} \\ \frac{n-1}{2}, & \text{when } n \text{ is odd} \end{cases}$$

For instance, when $n = 13$, n is odd, so $C = \frac{13-1}{2} = 6$. This type of description of a function is called a **piecewise definition** because it breaks the domain into pieces, and there is a rule for each piece.

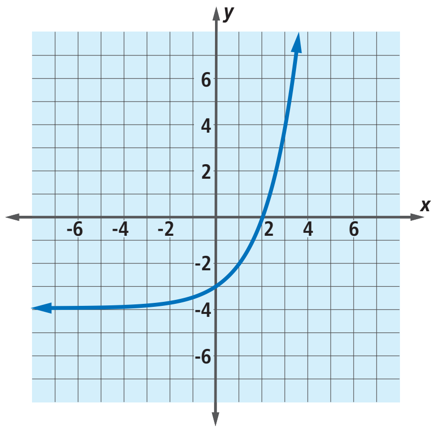
▸ QY2

In Example 2, for what value(s) of n does $C = 7$?

Often the domain and range of a function can be determined solely from a graph or an equation.

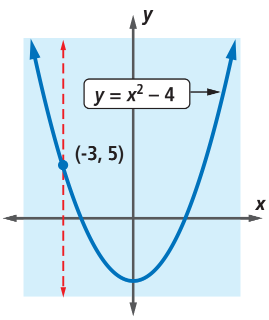
Example 3

A rule for the function graphed at the right is $y = 2^x - 4$. Find the domain and range of the function.

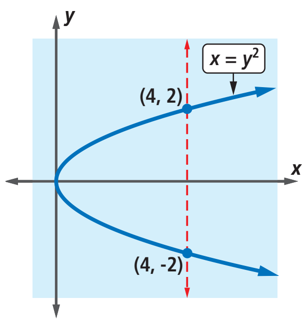


Solution The domain is the set of *x*-values for which $2^x - 4$ is defined, which is the set $\mathbb{R}$ of all real numbers. From the graph, the range appears to be the set of all real numbers greater than –4, which can be written as $\{y \mid y > -4\}$.

In a function, there is only one member of the range paired with each member of the domain. So, in a graph using rectangular coordinates, if *y* is a function of *x*, no vertical line will intersect the graph at more than one point. This is often referred to as the *vertical line test* for determining whether a relation is a function. You can see how this works on the two graphs of relations shown below. Only the relation graphed at the left is a function.



y is a function of x: no vertical line intersects the graph more than once.



y is not a function of x: there is at least one vertical line that intersects the graph more than once.

Naming Functions and Their Values

A function is usually named by a single letter such as *f* or *g*. For a function *f*, the symbol $f(x)$, which is read "*f* of *x*," indicates the value of the dependent variable when the independent variable is *x*. $f(x)$ is also called the **value** of the function at *x*. This symbol was first used by the mathematician Leonhard Euler (pronounced "oiler") in the 18th century. Euler's notation is particularly useful when evaluating functions at specific values of the independent variable. This notation is also used when defining functions on a computer algebra system (CAS).

Example 4

Suppose f is the function defined by the rule $f(x) = 4 \cdot \left(\frac{1}{2}\right)^x$ for all real numbers x.

a. Evaluate $f(5)$.

b. Does $f(-2 + 3) = f(-2) + f(3)$?

c. Evaluate $f(q + 1)$.

Solution 1 Use paper and pencil.

a. Substitute 5 for x: $f(5) = 4 \cdot \left(\frac{1}{2}\right)^5 = 4 \cdot \frac{1}{32} = \frac{1}{8}$.

b. Evaluate the left side. Work within parentheses first.

$$f(-2 + 3) = f(1) = 4 \cdot \left(\frac{1}{2}\right)^1 = 2$$

Evaluate the right side.

$$f(-2) + f(3) = 4\left(\frac{1}{2}\right)^{-2} + 4\left(\frac{1}{2}\right)^3 = 16 + \frac{1}{2} = 16\frac{1}{2}$$

So, $f(-2 + 3) \neq f(-2) + f(3)$.

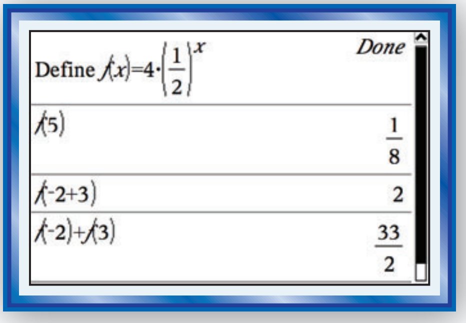
c. Substitute $q + 1$ for x in the rule.

$$f(q + 1) = 4 \cdot \left(\frac{1}{2}\right)^{q+1}$$

Solution 2 Define the function f on a CAS.

a. Input $f(5)$.

b. Input $f(-2 + 3)$. Input $f(-2) + f(3)$. The screenshot verifies the calculations in Solution 1.



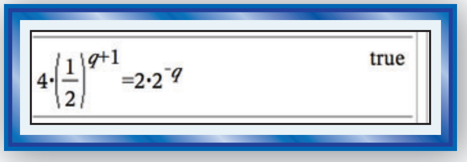
c. The screenshot at the right indicates that $f(q + 1) = 2 \cdot 2^{-q}$. Yet our answer in Solution 1 was $f(q + 1) = 4 \cdot \left(\frac{1}{2}\right)^{q+1}$. To show that these two forms are equivalent, convert one answer to the other. We write 4 and $\frac{1}{2}$ as powers of 2.

$$4 \cdot \left(\frac{1}{2}\right)^{q+1} = 2^2 \cdot (2^{-1})^{q+1} \qquad x^{-n} = \frac{1}{x^n}$$

$$= 2^2 \cdot 2^{-q-1} \qquad \text{Power of a Power Property}$$

$$= 2^1 \cdot 2^{-q} \qquad \text{Product of Powers Property}$$

The result can be verified using a CAS.



Part b of Example 4 illustrates that, in general, $f(a + b) \neq f(a) + f(b)$. That is, there is no general distributive property for functions over addition.

Questions

COVERING THE IDEAS

In 1–3, identify the independent variable and the dependent variable.

1. A parent bases a child's allowance on the number of chores completed by the child.
2. The participation grade in a class is calculated, in part, by the student's attendance.
3. Trees grow from sunlight and water.

In 4 and 5, give a definition of the term.

4. domain
5. range

6. An online photo lab usually charges \$0.25 per print to make a color print from a digital file. During a special promotion, customers receive a \$0.50 discount if 12 or more prints are made.
   a. Which is true, "the cost c is a function of the number n of prints made" or "the number n of prints made is a function of the cost c?"
   b. What is the cost of making 20 color prints from digital files?
   c. List all the ordered pairs (n, c) for $0 \le n \le 16$.
   d. Graph the relation in Part c for $0 \le n \le 16$.
   e. Write a piecewise formula for c in terms of n.

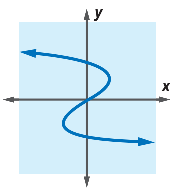
7. Consider the function defined by $y = f(x)$. What symbol represents each of the following?
   a. the function
   b. the dependent variable
   c. the value of the function
   d. the independent variable

8. Let g be the function defined by $g(t) = t^2 - 5$.
   a. Compute $g(-7)$.
   b. Find the value(s) of t such that $g(t) = 12$.
   c. Find the domain of g.
   d. Find the range of g.
   e. Evaluate $g(p + 3)$.

9. Consider $h(x) = \sqrt{x + 3}$.
   a. Evaluate $h(q - 1)$.
   b. For what value of q does $h(q - 1) = 3$?

In 10–12, a relation is graphed on a rectangular coordinate grid. Tell whether the relation is a function.

10.



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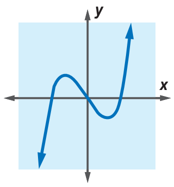

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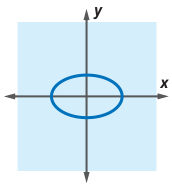
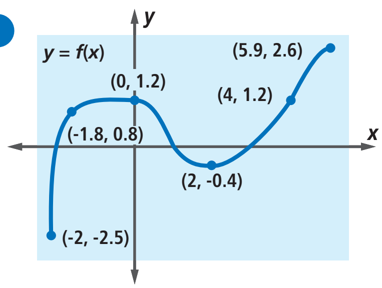


APPLYING THE MATHEMATICS

13. Refer to the graph of $y = f(x)$ at the right.
    a. Determine the domain and range of f.
    b. Find $f(2)$.
    c. When does $f(x) = 1.2$?



14. Refer to the national debt model, $y = 0.814(1.093)^x$, on page 79.

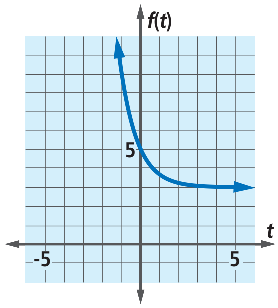
a. According to the model, what was the national debt in 1995? How close is this to the actual value?

b. According to the graph of the model, in what year was the national debt first one trillion dollars?

In 15–17, consider the relation defined by each sentence. Sketch a graph. Tell if the sentence defines a function. If so, give its domain and range.

15. $y = 3 \cdot 2^x$ 16. $y = -\sqrt{x}$ 17. $y < x + 1$

18. At the right is the graph of a function whose equation is $f(t) = 3 + 2 \cdot (3)^{-t}$.



a. Find the domain and range of the function.

b. Use the graph to approximate $f\left(\frac{1}{2}\right)$.

c. Use the graph to estimate the value of t such that $f(t) = 3.5$.

REVIEW

In 19 and 20, suppose that six used cars of a particular make and model are advertised in the newspaper for the following prices: \$14,950; \$15,250; \$14,500; \$14,700; \$14,250; \$14,900. (Lessons 1-6, 1-2)

19. Let c_i = the cost of the ith used car advertised.

a. Evaluate $\frac{\sum_{i=1}^{6} c_i}{6}$.

b. Which statistical measure is represented by the quantity in Part a?

20. a. Without calculating, tell why the standard deviation of this data set will be less than 500.

b. Calculate the standard deviation to verify your answer to Part a.

21. ***Skill Sequence*** Find the missing expression. **(Previous Course)**

a. $x^2 - 12x + \underline{\ ?\ } = (x - 6)^2$ b. $x^2 + \underline{\ ?\ } + 25 = (x + 5)^2$

c. $x^2 + 22x + 121 = (\underline{\ ?\ })^2$ d. $x^2 + 2ax + a^2 = (\underline{\ ?\ })^2$

22. Without graphing, determine whether the point (3, -4) is on the line with equation $y = 3x - 5$. **(Previous Course)**

23. Write an equation for the line with slope $-\frac{3}{2}$ that passes through (-6, 4). **(Previous Course)**

EXPLORATION

24. Find out the size of the U.S. national debt for a date as close as possible to today's date. What does the equation on page 79 predict for the national debt on the date you have? Calculate the percent error of the prediction.

QY ANSWERS

1. domain {5, 6, 7, 8}; range {1, 3}.

2. $n = 14$ and $n = 15$